

# THE KNESER-TITS CONJECTURE FOR GROUPS WITH TITS-INDEX $E_{8,2}^{66}$ OVER AN ARBITRARY FIELD

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**ABSTRACT.** We prove: (1) The group of multipliers of similitudes of a 12-dimensional anisotropic quadratic form over a field  $K$  with trivial discriminant and split Clifford invariant is generated by norms from quadratic extensions  $E/K$  such that  $q_E$  is hyperbolic. (2) If  $G$  is the group of  $K$ -rational points of an absolutely simple algebraic group whose Tits index is  $E_{8,2}^{66}$ , then  $G$  is generated by its root groups, as predicted by the Kneser-Tits conjecture.

## 1. INTRODUCTION

The Kneser-Tits conjecture—first formulated in [21]—predicts that the group of  $K$ -rational points (for some field  $K$  of arbitrary characteristic) of an absolutely simple algebraic group with Tits index



is generated by its root groups. This Tits-index is denoted by  $E_{8,2}^{66}$  in [19]. Groups with this Tits index are classified by similarity classes of anisotropic 12-dimensional quadratic forms over  $K$  with trivial discriminant and split Clifford invariant. By [22, 42.6], they are also the groups whose corresponding spherical building is a Moufang quadrangle of type  $E_8$  as defined in [22, 16.6].

Given a quadratic form  $q$  defined over a field  $K$ , we denote by  $\text{clif}(q)$  the Clifford invariant of  $q$ , by  $G(q)$  the group of multipliers of similitudes of  $q$ , by  $\text{Hyp}(q)$  the subgroup of  $K^\times$  generated by  $K^{\times 2}$  and the norms from finite extensions  $E/K$  such that  $q_E$  is hyperbolic and by  $\text{Hyp}_2(q)$  the subgroup of  $\text{Hyp}(q)$  generated by  $K^{\times 2}$  and the norms from *quadratic* extensions  $E/K$  such that  $q_E$  is hyperbolic (including inseparable ones).

Our goal is to prove the following closely related statements.

**Theorem 1.1.** *If  $q$  is an anisotropic quadratic form with trivial discriminant, then  $G(q) = \text{Hyp}_2(q)$  in the following cases:*

- (i)  $\dim q = 8$  and the index of  $\text{clif}(q)$  is 2;
- (ii)  $\dim q = 12$  and  $\text{clif}(q)$  is split.

**Theorem 1.2.** *If  $G$  is the group of  $K$ -rational points of an absolutely simple algebraic group whose Tits index is  $E_{8,2}^{66}$ , then  $G$  is generated by its root groups.*

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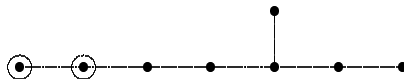
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We give two very different proofs of these theorems. In §2 we lay the groundwork that is common to the two proofs, and show that the equality  $G(q) = \text{Hyp}(q)$  holds for quadratic forms as in Theorem 1.1. As a consequence, the connected component of the identity  $\text{PGO}_+(q)$  in the group of projective similitudes is  $R$ -trivial if  $\text{char}(K) \neq 2$ : see Corollary 2.19. In §3, we give proofs of Theorems 1.1 and 1.2 based on results in [23] and [24]. In particular, the notion of a *quadrangular algebra* introduced in Chapters 12–13 of [22] and in [23] plays a central role in these proofs. In §4 we show how the  $R$ -triviality of  $\text{PGO}_+(q)$  for  $q$  as in Theorem 1.1(ii) in characteristic 0 yields another proof of Theorem 1.2 in arbitrary characteristic. In §5 we give an entirely different proof of Theorem 1.1 under the assumption that  $\text{char}(K) \neq 2$ , using the triality-defined correspondence between 8-dimensional quadratic forms of trivial discriminant and hermitian forms over the simple components of their even Clifford algebra.

For a survey of what is known about the Kneser-Tits conjecture; see [9]. We call attention especially to §6 of that paper, where the Kneser-Tits conjecture over an arbitrary field is discussed. By [17], [9, 6.1] and Theorem 1.2, the only exceptional groups of relative rank at least 2 for which the Kneser-Tits conjecture remains to be verified over arbitrary fields are those whose Tits index is



(called  $E_{8,2}^{78}$  in [19]). Groups with this Tits index are classified by isotopy classes of Albert division algebras, and the corresponding spherical buildings are the Moufang hexagons defined in [22, 16.8] for “hexagonal systems” of dimension 27. See also [9, 8.6] and [22, 37.41].

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## 2. SIMILITUDES OF QUADRATIC FORMS

Our main background reference for quadratic forms is [6], although we mostly use the notation of [23]. Let  $(K, L, q)$  be a quadratic space. Thus  $K$  is a field,  $L$  is a  $K$ -vector space and  $q: L \rightarrow K$  is a quadratic form on  $L$ . We let  $f = \partial q$  denote the polar bilinear form of  $q$ . Thus

$$f(x, y) = q(x + y) - q(x) - q(y) \quad \text{for } x, y \in L.$$

The quadratic space  $(K, L, q)$  is *nondegenerate* if  $\dim_K \text{rad } f \leq 1$ ; see [6, 7.17]. If  $\dim_K L$  is even and  $(K, L, q)$  is nondegenerate, then  $f$  is nondegenerate, the discriminant  $\text{disc}(q)$  is the isomorphism class of the center of the even Clifford algebra  $C_0(q)$  and the Clifford invariant  $\text{clif}(q)$  is the Brauer class of the full Clifford algebra  $C(q)$ ; see [6, §§13, 14]. As in [6, §§8, 9], we let  $I_q K$  denote the quadratic Witt group of  $K$  and let  $I_q^n K = I^{n-1} K \cdot I_q K$  for all  $n > 0$ , where  $I^{n-1} K$  is the  $(n-1)$ st power of the fundamental ideal  $IK$  of even-dimensional forms in the bilinear Witt ring  $WK$ .

The following definitions are taken from [22, 21.31].

**Definition 2.1.** A quadratic space  $(K, L, q)$  is of type  $E_7$  if it is anisotropic and there exists a separable quadratic extension  $E/K$  with norm  $N$  and scalars  $\alpha_1, \dots, \alpha_4$  such that

$$(K, L, q) \cong (K, E^4, \alpha_1 N \perp \alpha_2 N \perp \alpha_3 N \perp \alpha_4 N)$$

and

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \notin N(E).$$

In other words,  $q$  is anisotropic,

$$q \cong \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cdot N,$$

and the quaternion algebra  $(E/K, \alpha_1 \alpha_2 \alpha_3 \alpha_4)$ , which represents  $\text{clif}(q)$ , is not split.

**Definition 2.2.** A quadratic space  $(K, L, q)$  is of type  $E_8$  if it is anisotropic and there exists a separable quadratic extension  $E/K$  with norm  $N$  and scalars  $\alpha_1, \dots, \alpha_6$  such that

$$(K, L, q) \cong (K, E^6, \alpha_1 N \perp \alpha_2 N \perp \dots \perp \alpha_6 N)$$

and

$$-\alpha_1 \alpha_2 \dots \alpha_6 \in N(E).$$

In other words,  $q$  is anisotropic,

$$q \cong \langle \alpha_1, \dots, \alpha_6 \rangle \cdot N,$$

and  $\text{clif}(q)$  is split.

**Proposition 2.3.** *Suppose that  $(K, L, q)$  is an anisotropic quadratic space. Then the following hold:*

- (i)  $(K, L, q)$  is of type  $E_7$  if and only if  $\dim q = 8$ ,  $\text{disc}(q)$  is trivial and  $\text{clif}(q)$  is of index 2.
- (ii)  $(K, L, q)$  is of type  $E_8$  if and only if  $\dim q = 12$ ,  $\text{disc}(q)$  is trivial and  $\text{clif}(q)$  is split. These conditions are also equivalent to  $\dim q = 12$  and  $q \in I_q^3 K$ .

*Proof.* If  $\text{char}(K) \neq 2$ , (i) is in [11, Ex. 9.12] and (ii) in [16, p. 123]. In arbitrary characteristic, see [5, 4.12] and (for the second part of (ii)) [6, Thm. 16.3] if  $\text{char}(K) = 2$ .  $\square$

**Remark 2.4.** Suppose that  $(K, L, q)$  is a quadratic space of type  $E_7$  and that  $q(1) = 1$  for a distinguished element 1 of  $L$ . Let  $E$  and  $\alpha_1, \dots, \alpha_4$  be as in 2.1. By [23, 2.24],

$$(2.5) \quad C(q, 1) \cong M(4, D) \oplus M(4, D),$$

where  $C(q, 1)$  is the Clifford algebra with base point as defined in [22, 12.47] and  $D$  is the quaternion division algebra  $(E/K, \alpha_1 \alpha_2 \alpha_3 \alpha_4)$ . By [22, 12.51],  $C(q, 1)$  is isomorphic to the even Clifford algebra  $C_0(q)$ . Since  $D$  represents  $\text{clif}(q)$ , it is independent of the choice of the orthogonal decomposition of  $q$  in 2.1.

Our goal in this section is to prove the equality  $G(q) = \text{Hyp}(q)$  for  $q$  of type  $E_7$  or  $E_8$ . We start with some general observations. The following is essentially [23, 2.18].

**Proposition 2.6.** *Suppose that the polynomial  $p(x) = x^2 - \alpha x + \beta \in K[x]$  is separable and irreducible over  $K$ . Let  $E$  be the splitting field of  $p(x)$  over  $K$  and let  $N$  denote the norm of the extension  $E/K$ , so that  $(K, E, N)$  is a nondegenerate anisotropic 2-dimensional quadratic space. Let  $(K, L, q)$  be a finite-dimensional quadratic space. Then the following assertions are equivalent:*

- (i) *The  $K$ -vector space structure on  $L$  extends to an  $E$ -vector space structure such that  $q(u \cdot v) = N(u)q(v)$  for all  $u \in E, v \in L$ ;*
- (ii) *There exists a similitude  $T$  of  $q$  such that  $q(T(v)) = \beta q(v)$  and  $f(v, T(v)) = \alpha$  for all non-zero  $v \in L$  and  $p(T) = 0$ ;*
- (iii) *For each  $v_1 \in L$  there exists a decomposition  $L = V_1 \oplus \cdots \oplus V_d$  for some  $d \in \mathbb{N}$  such that  $v_1 \in V_1$ , the restriction  $q_i$  of  $q$  to  $V_i$  is similar to  $N$  for each  $i \in [1, d]$  and  $q = q_1 \perp \cdots \perp q_d$ ;*
- (iv) *for some  $d \in \mathbb{N}$  and some  $\alpha_1, \alpha_2, \dots, \alpha_d \in K^\times$ ,*

$$(K, L, q) \cong (K, E^d, \alpha_1 N \perp \alpha_2 N \perp \cdots \perp \alpha_d N);$$

- (v)  *$q_E$  is hyperbolic.*

*Proof.* Suppose that (i) holds, choose a root  $\gamma \in E$  of  $p(x)$  and let  $T(v) = \gamma \cdot v$  for all  $v \in L$ . Then  $T$  is a similitude of  $q$  as in (ii). If  $T$  is a similitude of  $q$  as in (ii), then for each nonzero  $v \in L$  the restriction of  $q$  to  $\langle v, T(v) \rangle$  is similar to  $N$  (and, in particular, is nondegenerate). Therefore (iii) holds, and (iii) of course implies (iv). Fixing an isomorphism as in (iv), we may transfer to  $L$  the natural  $E$ -vector space structure on  $E^d$  to obtain (i). The equivalence of (iv) and (v) follows readily from [6, Prop. 34.8].  $\square$

**Definition 2.7.** A similitude  $\varphi$  of a quadratic space  $(K, L, q)$  is called *inseparable* if  $\text{char}(K) = 2$ , the multiplier of  $\varphi$  is not in  $K^{\times 2}$  and

$$f(v, \varphi(v)) = 0 \quad \text{for all } v \in L,$$

where  $f = \partial q$ . We call a similitude of  $q$  *separable* if it is not inseparable. Thus, if  $\text{char}(K) \neq 2$  all similitudes are separable.

**Proposition 2.8.** *Let  $(K, L, q)$  be a finite-dimensional quadratic space such that  $f = \partial q$  is nondegenerate. If  $(K, L, q)$  admits an inseparable similitude with multiplier  $\gamma$ , then*

$$q \simeq \langle 1, \gamma \rangle \cdot q_0$$

*for some non-degenerate quadratic form  $q_0$ . In particular,  $\dim L \equiv 0 \pmod{4}$  and  $q_{K(\sqrt{\gamma})}$  is hyperbolic.*

*Proof.* Let  $E = K(\sqrt{\gamma})$  be a purely inseparable quadratic extension of  $K$ , and let  $\varphi$  be an inseparable similitude of  $(K, L, q)$  with multiplier  $\gamma$ . Linearizing the condition  $f(v, \varphi(v)) = 0$ , we obtain

$$f(v, \varphi(w)) = f(\varphi(v), w) \quad \text{for all } v, w \in L.$$

Since  $f(\varphi(v), \varphi(w)) = \gamma f(v, w)$  for all  $v, w \in L$ , it follows that

$$f(v, \varphi^2(w)) = f(\varphi(v), \varphi(w)) = \gamma f(v, w) \quad \text{for all } v, w \in L,$$

hence  $\varphi^2(w) = \gamma w$  for all  $w \in L$ . We then define on  $L$  an  $E$ -vector space structure by

$$(\lambda + \mu\sqrt{\gamma}) \cdot v = \lambda v + \mu\varphi(v) \quad \text{for } \lambda, \mu \in K \text{ and } v \in L,$$

and we define a map  $f': L \times L \rightarrow E$  by

$$f'(v, w) = f(v, w) + \sqrt{\gamma^{-1}}f(v, \varphi(w)) \quad \text{for } v, w \in L.$$

A straightforward computation shows that  $f'$  is a bilinear alternating form on  $L$ . It is nondegenerate since  $f$  is nondegenerate. Therefore, the dimension of  $L$  over  $E$  is even, hence its dimension over  $K$  is a multiple of 4. Let  $(e_i, e'_i)_{i=1}^d$  be a symplectic  $E$ -base of  $L$  for  $f'$ . If  $L_0 \subset L$  is the  $K$ -span of  $(e_i, e'_i)_{i=1}^d$  and  $q_0$  is the restriction of  $q$  to  $L_0$ , we have  $L = L_0 \perp \varphi(L_0)$  and  $q = q_0 \perp \langle \gamma \rangle q_0$ .  $\square$

**Corollary 2.9.** *Let  $(K, L, q)$  be a finite-dimensional quadratic space such that  $\partial q$  is non-degenerate. Then the multiplier of every inseparable similitude of  $q$  is in  $\text{Hyp}_2(q)$ .*

*Proof.* Let  $\gamma$  be the multiplier of an inseparable similitude of  $q$ . Clearly,  $\gamma \in N(K(\sqrt{\gamma}))$ , and Proposition 2.8 shows that  $q$  is hyperbolic over  $K(\sqrt{\gamma})$ .  $\square$

We now consider quadratic forms of low dimension. The following result is presumably well-known:

**Proposition 2.10.** *Every 10-dimensional quadratic form in  $I_q^3(K)$  is isotropic.*

*Proof.* This was proved by Pfister [16, p. 123] under the hypothesis that  $\text{char}(K) \neq 2$ . The arguments also apply when  $\text{char}(K) = 2$ ; see [5, Thm. 4.10].  $\square$

We now consider quadratic spaces of type  $E_7$ . For the next statement, we do not require the form to be anisotropic.

**Lemma 2.11.** *Let  $(K, L, q)$  be a nondegenerate quadratic space of dimension 8. If  $\text{disc}(q)$  is trivial and  $\text{clif}(q)$  is represented by a quaternion algebra  $Q$  with norm form  $N_Q$ , then  $q$  is Witt-equivalent to the sum of a multiple of  $N_Q$  and a multiple of some 3-fold Pfister quadratic form  $\pi$ : there exist  $\alpha, \beta \in K^\times$  such that*

$$(2.12) \quad q = \langle \alpha \rangle \cdot N_Q + \langle \beta \rangle \cdot \pi \quad \text{in } I_q K.$$

Moreover,  $G(q) = G(N_Q) \cap G(\pi)$ .

*Proof.* Let  $\alpha \in K^\times$  be a value represented by  $q$ . Consider the form

$$q' = q \perp \langle -\alpha \rangle \cdot N_Q.$$

This 12-dimensional form is isotropic and has trivial discriminant and Clifford invariant, hence it is in  $I_q^3 K$  and is Witt-equivalent to a 10-dimensional form. By Proposition 2.10, it is actually equivalent to an 8-dimensional form. By the Arason–Pfister Hauptsatz [6, Thm. 23.7], this 8-dimensional quadratic form becomes hyperbolic over the function field of the corresponding quadric, hence it is a multiple of some 3-fold Pfister quadratic form  $\pi$  by [6, Cor. 23.4]. Letting  $q' = \langle \beta \rangle \cdot \pi$  in  $I_q K$ , we have (2.12).

Now, for  $\gamma \in G(q)$  we have  $\langle 1, -\gamma \rangle \cdot q = 0$  in  $I_q K$ , hence

$$\langle 1, -\gamma \rangle \cdot \langle \alpha \rangle \cdot N_Q = -\langle 1, -\gamma \rangle \cdot \langle \beta \rangle \cdot \pi \quad \text{in } I_q K.$$

Since the left side is a form of dimension 8 and the right side is a form of dimension 16, the right side must be isotropic. It is then hyperbolic by [6, Cor. 9.10], since it is a multiple of a Pfister form. The left side is then also hyperbolic, which means that  $\gamma$  is in  $G(\pi)$  and in  $G(N_Q)$ . We have thus proved  $G(q) \subset G(N_Q) \cap G(\pi)$ . Since the reverse inclusion is clear, the proof is complete.  $\square$

Lemmas 2.13 and 2.14 are well-known when  $\text{char}(K) \neq 2$ ; see [7, 2.13] for Lemma 2.13. The proofs we give below do not require any separability hypothesis.

**Lemma 2.13.** *Let  $E_1, E_2$  be linearly disjoint quadratic extensions of a field  $K$ , and let  $M = E_1 \otimes_K E_2$ . The norm groups of  $E_1, E_2$ , and  $M$  are related as follows:*

$$N(E_1/K) \cap N(E_2/K) = K^{\times 2} \cdot N(M/K).$$

*Proof.* Since  $K^{\times 2} \subset N(E_i/K)$  and  $N(M/K) \subset N(E_i/K)$  for  $i = 1, 2$ , the inclusion  $N(E_1/K) \cap N(E_2/K) \supset K^{\times 2} \cdot N(M/K)$  is clear, and it suffices to prove the reverse inclusion. We identify  $E_1$  and  $E_2$  with subfields of  $M$  and consider  $\alpha \in N(E_1/K) \cap N(E_2/K)$ . Let  $x_1 \in E_1^\times, x_2 \in E_2^\times$  be such that

$$\alpha = N_{E_1/K}(x_1) = N_{E_2/K}(x_2).$$

Let  $T_{E_i/K}: E_i \rightarrow K$  be the trace map, for  $i = 1, 2$ . Computation shows that

$$x_1 N_{M/E_1}(1 + x_1^{-1} x_2) = T_{E_1/K}(x_1) + T_{E_2/K}(x_2).$$

If  $x_1 \neq -x_2$ , the left side is nonzero. Taking the norm from  $E_1$  to  $K$  of each side yields

$$\alpha N_{M/K}(1 + x_1^{-1} x_2) = (T_{E_1/K}(x_1) + T_{E_2/K}(x_2))^2 \in K^{\times 2},$$

hence  $\alpha \in K^{\times 2} \cdot N(M/K)$ . If  $x_1 = -x_2$ , then  $x_1 \in E_1 \cap E_2 = K$ , hence  $\alpha \in K^{\times 2}$ .  $\square$

**Lemma 2.14.** *Any multiplier of similitude of an anisotropic quadratic Pfister space  $(K, L, \pi)$  is a square in  $K$  or is the norm of a quadratic extension over which  $\pi$  is hyperbolic.*

*Proof.* By [6, Cor. 9.9], the multipliers of  $\pi$  are the represented values of  $\pi$ , so any  $\gamma \in G(\pi)$  has the form  $\gamma = \pi(v)$  for some  $v \in L$ . Let  $e \in L$  be such that  $\pi(e) = 1$ . If  $e$  and  $v$  are not linearly independent, then  $\gamma \in K^{\times 2}$ . Otherwise, let  $V$  be the  $K$ -span of  $e$  and  $v$ . The restriction of  $\pi$  to  $V$  is the norm form of a quadratic extension of  $K$  over which  $\pi$  is isotropic, hence hyperbolic by [6, Cor. 9.10]. By construction, this norm form represents  $\gamma$ .  $\square$

**Proposition 2.15.** *For any nondegenerate quadratic space  $(K, L, q)$  of dimension 8 such that  $\text{disc}(q)$  is trivial and  $\text{clif}(q)$  has index 1 or 2, we have  $G(q) = \text{Hyp}(q)$ .*

*Proof.* It suffices to show  $G(q) \subset \text{Hyp}(q)$ , since the reverse inclusion follows from the similarity norm principle [6, Thm. 20.14]. Let  $\gamma \in G(q)$ , and consider a decomposition of  $q$  as in (2.12). We may assume  $q$  is not hyperbolic, otherwise  $\text{Hyp}(q) = K^\times = G(q)$  and there is nothing to prove. If  $N_Q$  or  $\pi$  is isotropic, hence hyperbolic, the proposition readily follows from Lemma 2.14. For the rest of the proof, we may thus assume  $N_Q$  and  $\pi$  are anisotropic. By Lemma 2.11 we have  $\gamma \in G(N_Q) \cap G(\pi)$ , hence Lemma 2.14 yields quadratic extensions  $E_1, E_2$  of  $K$  that split  $N_Q$  and  $\pi$  respectively, such that  $\gamma \in N(E_1/K) \cap N(E_2/K)$ . If  $E_1 \cong E_2$ , then  $E_1$  splits  $N_Q$  and  $\pi$ , hence also  $q$ . Since  $\gamma \in N(E_1/K)$ , it follows that  $\gamma \in \text{Hyp}(q)$ . If  $E_1 \not\cong E_2$ , then  $E_1$  and  $E_2$  are linearly disjoint over  $K$ , and the tensor product  $M = E_1 \otimes_K E_2$  is a field that splits  $N_Q$  and  $\pi$ , hence also  $q$ . Since  $\gamma$  is a norm from  $E_1$  and from  $E_2$ , Lemma 2.13 shows that  $\gamma \in K^{\times 2} \cdot N(M/K)$ , hence  $\gamma \in \text{Hyp}(q)$ .  $\square$

Proposition 2.15 applies in particular to quadratic spaces of type  $E_7$ . We now turn to spaces of type  $E_8$ .

**Proposition 2.16.** *Suppose that  $(K, L, q)$  is of type  $E_8$ , that  $\gamma$  is the multiplier of a separable similitude of  $q$  and that  $\gamma \notin K^{\times 2}$ . Then there exists a decomposition*

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2)$$

*such that  $q_2$  is of type  $E_7$ ,  $q_1$  is similar to the reduced norm of the quaternion division algebra representing  $\text{clif}(q_2)$  and  $\gamma$  is a multiplier of similitudes of both  $q_1$  and  $q_2$ .*

*Proof.* Let  $\varphi$  be a separable similitude of  $q$  with multiplier  $\gamma$ . If  $\text{char}(K) = 2$ , we choose  $v \in L$  such that  $f(v, \varphi(v)) \neq 0$ ; if  $\text{char}(K) \neq 2$ , we let  $v$  be an arbitrary non-zero vector in  $L$ . Next we set  $W = \langle v, \varphi(v) \rangle$ . Since  $\gamma \notin K^{\times 2}$ , we have  $\dim_K W = 2$ . Let  $\hat{q}_1$  denote the restriction of  $q$  to  $W$  and let  $\hat{q}_2$  denote the restriction of  $q$  to  $W^\perp$ . The form  $\hat{q}_1$  is similar to the norm  $N$  of a quadratic extension  $E/K$  such that  $\gamma \in N(E)$ . Since  $\hat{q}_1$  is nondegenerate, the extension  $E/K$  is separable and  $q = \hat{q}_1 \perp \hat{q}_2$ . Since  $q_E$  and  $(\hat{q}_1)_E$  have trivial discriminant and split Clifford invariant, also  $(\hat{q}_2)_E$  has trivial discriminant and split Clifford invariant. By Proposition 2.10, it follows that  $(\hat{q}_2)_E$  is isotropic. Hence we can choose a 2-dimensional subspace  $U$  of  $W^\perp$  such that the restriction of  $\hat{q}_2$  to  $U$  is hyperbolic over  $E$ . By Proposition 2.6, the restriction of  $\hat{q}_2$  to  $U$  is similar to  $N$ . Let  $L_1 = W \oplus U$ , let  $L_2 = L_1^\perp$  and let  $q_i$  denote the restriction of  $q$  to  $L_i$  for  $i = 1$  and 2. Then  $q_1$  is similar to the reduced norm of a quaternion division algebra  $D$ ,  $\gamma$  is a multiplier of  $q_1$  and

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2).$$

Since  $\gamma$  is a multiplier of both  $q$  and  $q_1$ , both  $\langle 1, -\gamma \rangle \cdot q$  and  $\langle 1, -\gamma \rangle \cdot q_1$  are hyperbolic. Since

$$\langle 1, -\gamma \rangle \cdot q = \langle 1, -\gamma \rangle \cdot q_1 \perp \langle 1, -\gamma \rangle \cdot q_2,$$

it follows by Witt's Cancellation Theorem that the product  $\langle 1, -\gamma \rangle \cdot q_2$  is also hyperbolic. Hence  $q_2 = \langle \gamma \rangle \cdot q_2$  in  $I_q K$ . By [6, 8.17], therefore, there is a similitude of  $q_2$  with multiplier  $\gamma$ . Since  $\text{disc}(q)$  and  $\text{disc}(q_1)$  are both trivial, so is  $\text{disc}(q_2)$ . Furthermore,  $\text{clif}(q_2) = \text{clif}(q_1)$  since  $\text{clif}(q)$  is split. Since the quaternion division algebra  $D$  represents  $\text{clif}(q_1)$ , it also represents  $\text{clif}(q_2)$ . We conclude, in particular, that  $q_2$  is of type  $E_7$ .  $\square$

**Corollary 2.17.** *For any nondegenerate quadratic space  $(K, L, q)$  of dimension 12 such that  $\text{disc}(q)$  and  $\text{clif}(q)$  are trivial, we have  $G(q) = \text{Hyp}(q)$ .*

*Proof.* As in Proposition 2.15, it suffices to prove  $G(q) \subset \text{Hyp}(q)$ . If  $q$  is isotropic, then Proposition 2.10 shows that  $q$  is Witt-equivalent to a multiple of a 3-fold Pfister form, hence the inclusion follows from Lemma 2.14. For the rest of the proof, we may thus assume  $q$  is anisotropic, i.e.,  $q$  is of type  $E_8$ .

Let  $\gamma \in G(q)$ . If  $\gamma$  is the multiplier of an inseparable similitude, then we have  $\gamma \in \text{Hyp}(q)$  by Corollary 2.9. If  $\gamma$  is the multiplier of a separable similitude, we fix a decomposition  $q = q_1 \perp q_2$  as in Proposition 2.16, so  $\gamma \in G(q_1) \cap G(q_2)$ . By Proposition 2.15 we have  $G(q_2) = \text{Hyp}(q_2)$ . Now, if  $E/K$  is a finite extension such that  $(q_2)_E$  is hyperbolic, then  $E$  splits  $\text{clif}(q_2)$ . Hence  $(q_1)_E$  is hyperbolic, and therefore  $q_E$  is hyperbolic. This shows  $\text{Hyp}(q_2) \subset \text{Hyp}(q)$ . Since  $\gamma \in \text{Hyp}(q_2)$ , it follows that  $\gamma \in \text{Hyp}(q)$ .  $\square$

**Remark 2.18.** Restricting to *quadratic* extensions that split  $q_2$  in the last part of the proof above, we see that  $\text{Hyp}_2(q_2) \subset \text{Hyp}_2(q)$ . This observation will be used in §5.

When  $\text{char}(K) \neq 2$ , Proposition 2.15 and Corollary 2.17 yield information on the connected component of the identity  $\text{PGO}_+(q)$  in the group of projective similitudes of  $(K, L, q)$ , which is the group of algebra automorphisms of  $\text{End}(L)$  that commute with the adjoint involution of  $q$ . The property of  $R$ -triviality used in the following statement refers to Manin's  $R$ -equivalence; see [14, §1] for details.

**Corollary 2.19.** *Assume  $\text{char}(K) \neq 2$ . For  $q$  a quadratic form of type  $E_7$  or  $E_8$  over  $K$ , the group  $\text{PGO}_+(q)$  is  $R$ -trivial.*

*Proof.* By [14, Thm. 1], it suffices to prove that  $G(q_E) = \text{Hyp}(q_E)$  for every field  $E$  containing  $K$ . If  $q$  is of type  $E_7$ , this property readily follows from Proposition 2.15. If it is of type  $E_8$ , it follows from Corollary 2.17.  $\square$

**Remark 2.20.** Skip Garibaldi has observed that if  $q$  is of type  $E_7$ , then it follows from Lemma 2.11 and [8, Prop. 6.1] that the group  $\text{PGO}_+(q)$  is actually stably rational.

### 3. QUADRANGULAR ALGEBRAS AND PROOFS OF THEOREMS 1.1 AND 1.2

Most of this section is devoted to results about quadrangular algebras. At the very end of this section, we use these results to prove Theorems 1.1 and 1.2.

The notion of a quadrangular algebra arose in the course of the classification of Moufang polygons; see, in particular, Chapters 12–13 and 27 in [22]. For the definition, see [23, 1.17].

**Proposition 3.1.** *Let  $(K, L, q)$  be a quadratic space of type  $E_7$  or  $E_8$  as defined in 2.1 and 2.2 and suppose that  $q(1) = 1$  for a distinguished element 1 of  $L$ . Then there exists a unique quadrangular algebra*

$$\Xi = (K, L, q, 1, X, \cdot, h, \theta)$$

as defined in [23, 1.17].

*Proof.* Existence holds by [23, Thm. 10.1] and uniqueness (up to equivalence as defined in [23, Thm. 1.22]) holds by [23, 6.42].  $\square$

**Notation 3.2.** For the rest of this section, we let

$$\Xi = (K, L, q, 1, X, \cdot, h, \theta)$$

be as in 3.1 and  $f = \partial q$ . By [23, Prop. 4.2], we can assume that  $\Xi$  is  $\delta$ -standard for some  $\delta \in L$  as defined in [23, 4.1]. (This allows us to use the identities in Chapter 4 of [23].) In addition, we let  $\sigma$  be as [23, 1.2], we let  $u^{-1}$  for all non-zero  $u \in L$  be as in [23, 1.3] and we let  $\pi$  be as in [23, 1.17(D1)].

**Remark 3.3.** Suppose that  $(K, L, q)$  is of type  $E_7$  and let  $C(q, 1)$  and  $D$  be as in 2.4. By (2.5) and [23, 1.17(A1)–(A3) and Prop. 2.22], there exists a unique map  $*$  from  $D \times X$  to  $X$  with respect to which  $X$  is a left vector space over  $D$ ,  $ta = t * a$  for all  $(a, t) \in X \times K$  and  $w * (a \cdot v) = (w * a) \cdot v$  for all  $w \in D$ ,  $a \in X$  and  $v \in L$ . This map is given explicitly in [24, 3.6].



**Proposition 3.4.** *Suppose that  $(K, L, q)$  is of type  $E_7$ , let  $D$  and  $*$  be as in 3.3, let  $\varphi_1$  be a similitude of  $q$ , let  $u = \varphi_1(1)$  and let*

$$a \hat{\cdot} v = (a \cdot v) \cdot u^{-1}$$

*for all  $(a, v) \in X \times L$ , where  $u^{-1}$  is as in 3.2. Then there exists a similitude  $\varphi$  with the same multiplier as  $\varphi_1$  such that  $u = \varphi(1)$ , an element  $\omega \in K^\times$  and a  $D$ -linear automorphism  $\psi$  of  $X$  such that the following hold:*

- (i)  $\psi(a \cdot v) = \psi(a) \hat{\cdot} \varphi(v)$  for all  $a \in X$  and all  $v \in L$ .
- (ii)  $\varphi(h(a, b)) = \omega h(\psi(a), \psi(b)u)$  for all  $a, b \in X$ .
- (iii)  $\varphi(\theta(a, v)) \equiv \omega \theta(\psi(a), \varphi(v)) \pmod{\langle \varphi(v) \rangle}$  for all  $a \in X$  and all  $v \in L$ .

*Proof.* The map  $\varphi_1$  is an isomorphism of pointed quadratic spaces from  $(K, L, q, 1)$  to  $(K, L, q/q(u), u)$ . It therefore induces an isomorphism of Clifford algebras with base point from  $C(q, 1)$  to  $C(q/q(u), u)$ . Let  $\hat{\Xi}$ ,  $\hat{h}$  and  $\hat{\theta}$  be as in [23, Prop. 8.1]; thus,  $\hat{\Xi} = (K, L, q/q(u), u, X, \hat{\cdot}, \hat{h}, \hat{\theta})$  is the isotope of  $\Xi$  at  $u$  as defined in [23, 8.7]. By [23, 1.17(A1)–(A3) and Prop. 2.22] applied to both  $\Xi$  and to  $\hat{\Xi}$ ,  $X$  is a right  $C(q, 1)$ -module with respect to  $\cdot$  and a right  $C(q/q(u), u)$ -module with respect to  $\hat{\cdot}$ . By [22, 12.55] (where the base point 1 is called  $\epsilon$ ), exactly one of the two direct summands in (2.5) acts nontrivially on  $X$ , and by [22, 12.54], there exists an isometry  $\rho$  of  $q$  fixing 1 that extends to an automorphism of  $C(q, 1)$  interchanging the two direct summands. Thus for  $j = 0$  or  $1$ , the composition  $\varphi_1 \circ \rho^j$  maps the direct summand  $A$  of  $C(q, 1)$  acting nontrivially on  $X$  to the direct summand  $A_u$  of  $C(q/q(u), u)$  acting nontrivially on  $X$ . Let  $\varphi = \varphi_1 \circ \rho^j$ . Choosing a basis for  $X$  as a left vector space over  $D$ , we can identify both  $A$  and  $A_u$  with  $\text{End}_D(X)$ . It follows that there exists a  $D$ -linear automorphism  $\psi$  of  $X$  such that (i) holds. By [23, 1.25 and Prop. 6.38], there exists  $\omega \in K^\times$  such that also (ii) and (iii) hold.  $\square$

**Notation 3.5.** Let  $g$  and  $\phi$  be the maps that appear in [23, 1.17(C3)–(C4)], let

$$(U_+, U_1, U_2, U_3, U_4)$$

be the root group sequence and  $x_4$  the isomorphism from  $L$  to  $U_4$  obtained by applying the recipe in [22, 16.6] to  $\Xi$ ,  $g$  and  $\phi$ , let  $\Gamma$  be the corresponding Moufang quadrangle (see [22, 8.11]), let  $G^\dagger$  be the subgroup of  $\text{Aut}(\Gamma)$  generated by the root groups of  $\Gamma$ , let  $H_0$  be the subgroup of  $\text{Aut}(\Gamma)$  defined in [24, 1.4] (or [23, 11.20]), let  $G = H_0 \cdot G^\dagger$  and let  $H^\dagger = H_0 \cap G^\dagger$ . By [22, 35.11], the similarity class of  $(K, L, q)$  is an invariant of  $\Gamma$ .

**Proposition 3.6.**  $G/G^\dagger \cong H_0/H^\dagger$  and if  $(K, L, q)$  is of type  $E_8$ , then  $G$  is the group of  $K$ -rational points of an absolutely simple algebraic group with Tits index  $E_{8,2}^{66}$ , and every such group arises in this way starting with some quadratic space of type  $E_8$  defined over  $K$ .

*Proof.* For the isomorphism  $G/G^\dagger \cong H_0/H^\dagger$ , see the top of page 193 of [24] (where  $G$  is called  $G_0$ ). The remaining assertions hold by [22, 42.6].  $\square$

**Proposition 3.7.** *Suppose that  $(K, L, q)$  is of type  $E_7$  and that  $\varphi_1$  is a similitude of  $q$ . Let  $H_0$  and  $x_4$  be as in 3.5. Then there exist an element  $h$  of  $H_0$  and a similitude  $\varphi$  of  $q$  with the same multiplier as  $\varphi_1$  such that*

$$x_4(v)^h = x_4(\varphi(v))$$

*for all  $v \in L$ .*

*Proof.* Let  $\varphi$  and  $\psi$  be the maps obtained by applying Proposition 3.4 to  $\varphi_1$ . By Proposition 3.4 and [23, Prop. 12.5], the pair  $(\varphi, \psi)$  is contained in the structure group of  $\Xi$  (as defined in [23, 12.4]). The claim follows now by [23, Thm. 12.11] and the first few lines of its proof (as well as [23, 11.22]).  $\square$

From now on we identify  $K$  with its image under the map  $t \mapsto t \cdot 1$  from  $K$  to  $L$ . Thus when we write  $\pi(a) + t$  for  $(a, t) \in X \times K$ , for example, we mean  $\pi(a) + t \cdot 1$  (where  $\pi$  is as in 3.2).

**Proposition 3.8.** *Let  $a$  be a non-zero element of  $X$ , let*

$$p(x) = x^2 - f(1, \pi(a))x + q(\pi(a)) \in K[x],$$

*let  $E$  be the splitting field of  $p(x)$  over  $K$  and let  $N$  be the norm of the extension  $E/K$ . Let  $T(v) = \theta(a, v)$  for all  $v \in L$  and let  $I$  be the identity automorphism of  $L$ . Then  $N(E^\times) = K^{\times 2} \cdot \{q(\pi(a) + t) \mid t \in K\}$ ,  $q_E$  is hyperbolic and for each  $t \in K$ ,  $T + tI$  is a similitude of  $q$  with multiplier  $q(\pi(a) + t)$ .*

*Proof.* By [23, 1.17(D2)],  $p(t) = q(\pi(a) - t) \neq 0$  for each  $t \in K$ . Thus  $p(x)$  is irreducible over  $K$ . It follows that  $N(E^\times) = K^{\times 2} \cdot \{q(\pi(a) + t) \mid t \in K\}$ . By [23, Props. 4.9(i) and 4.22],  $T + tI$  is a similitude of  $q$  with multiplier  $q(\pi(a) + t)$  for each  $t \in K$  and  $f(T(v), v) = f(\pi(a), 1)q(v)$  for each non-zero  $v \in L$ . By [23, Prop. 4.21],  $p(T) = 0$ . Thus if  $p(x)$  is separable, then  $q_E$  is hyperbolic by Propositions 2.6. If  $p(x)$  is inseparable, then  $f(\pi(a), 1) = 0$ , hence  $T$  is inseparable and again  $q_E$  is hyperbolic, this time by Proposition 2.8.  $\square$

**Definition 3.9.** For each non-zero  $a \in X$ , the map  $v \mapsto \theta(a, v)$  is a similitude of  $q$  by Proposition 3.8. We call an element  $a \in X$  *separable* if  $a \neq 0$  and the similitude  $v \mapsto \theta(a, v)$  of  $q$  is separable as defined in 2.7. We let  $X_{\text{sep}}$  denote the set of separable elements of  $X$ . Thus if  $\text{char}(K) \neq 2$ , then  $X_{\text{sep}} = X \setminus \{0\}$ , but if  $\text{char}(K) = 2$ , then by [23, Prop. 4.9(i)],

$$X_{\text{sep}} = \{a \in X \mid f(\pi(a), 1) \neq 0\}.$$

If  $\text{char}(K) = 2$ , then by [22, 13.42–13.43],  $a \mapsto f(\pi(a), 1)$  is a nondegenerate quadratic form on  $X$ . In particular, the set  $X_{\text{sep}}$  is non-empty also if  $\text{char}(K) = 2$ .

**Proposition 3.10.** *Every inseparable similitude of  $q$  (as defined in 2.7) is the product of two separable similitudes.*

*Proof.* Let  $\varphi$  be an inseparable similitude of  $q$ , so  $\text{char}(K) = 2$ . It suffices to show that

$$f(\theta(a, \varphi(v)), v) \neq 0$$

for some  $v \in L$  and some  $a \in X_{\text{sep}}$ , where  $X_{\text{sep}}$  is as in 3.9. Suppose this is false and let  $w = \varphi(1)$ . Then

$$(3.11) \quad f(\theta(a, w), 1) = 0$$

for all  $a \in X_{\text{sep}}$ . Furthermore,

$$(3.12) \quad f(w, 1) = 0$$

but  $w \notin \langle 1 \rangle$  since  $\varphi$  is inseparable. Choose  $a \in X_{\text{sep}}$ . Since  $f$  is nondegenerate, we can choose  $v \in \langle w \rangle^\perp \setminus \langle 1 \rangle^\perp$ . Replacing  $v$  by  $v + w$  if necessary, we can assume in addition (by [23, Prop. 4.9(i)] again) that

$$(3.13) \quad f(\theta(a, w), v) \neq 0.$$

By [23, Prop. 3.21],  $av \in X_{\text{sep}}$ , so  $f(\theta(av, w), 1) = 0$  by (3.11). By [23, 1.17(C4)] and (3.12), it follows that

$$f(\theta(a, w^\sigma)^\sigma, 1)q(v) = f(w, v^\sigma)f(\theta(a, v)^\sigma, 1) + f(\theta(a, v), w^\sigma)f(v^\sigma, 1),$$

where  $\sigma$  is as in 3.2. By [23, 1.4] and (3.12), we have  $x^\sigma = x$  for  $x = 1$  and  $x = w$  and  $f(x^\sigma, y) = f(x, y^\sigma)$  for all  $x, y \in L$ . Therefore

$$f(\theta(a, w^\sigma)^\sigma, 1) = f(\theta(a, w), 1) = 0$$

by (3.11) and

$$f(w, v^\sigma) = f(w^\sigma, v) = f(w, v) = 0$$

by the choice of  $v$  and hence

$$f(\theta(a, v), w)f(v, 1) = f(\theta(a, v), w^\sigma)f(v^\sigma, 1) = 0.$$

Since  $f(v, 1) \neq 0$  by the choice of  $v$ , we conclude that  $f(\theta(a, v), w) = 0$ . By [23, Prop. 4.22], therefore,  $f(\theta(a, \theta(a, v)), \theta(a, w)) = 0$ . By [23, Prop. 4.21], it follows that

$$f(\pi(a), 1)f(\theta(a, v), \theta(a, w)) = q(\pi(a))f(v, \theta(a, w)).$$

By (3.13), therefore,  $f(\theta(a, v), \theta(a, w)) \neq 0$ . By one more application of [23, Prop. 4.22], however,  $f(\theta(a, v), \theta(a, w)) = q(\pi(a))f(v, w) = 0$ .  $\square$

**Proposition 3.14.** *Let  $E/K$  be a separable quadratic extension such that  $q_E$  is hyperbolic and let  $V_i$  and  $q_i$  for  $i \in [1, d]$  be as in Proposition 2.6(iii) with  $v_1 = 1$ . Then there exists  $e \in X_{\text{sep}}$  such that  $\theta(e, V_i) = V_i$  for each  $i \in [1, d]$ .*

*Proof.* Let  $p(x) = x^2 - \alpha x + \beta \in K[x]$  be an irreducible polynomial that splits over  $E$ . We can choose  $p(x)$  so that  $\alpha = 0$  if and only if  $\text{char}(K) \neq 2$ . Let  $\gamma, \gamma_1 \in E$  be the two roots of  $p(x)$ . There exists an  $E$ -vector space structure on  $L$  as in Proposition 2.6(i) such that  $V_i$  is a 1-dimensional subspace for each  $i \in [1, d]$ . Let  $T(v) = \gamma \cdot v$  for each  $v \in L$  and let  $T^\epsilon$  be the unique automorphism of  $L$  such that  $T^\epsilon(v) = \gamma_1 \cdot v$  for all  $v \in V_1$  and  $T^\epsilon(v) = T(v)$  for all  $v \in V_1^\perp$ . Both  $T$  and  $T^\epsilon$  are norm splitting maps of  $q$  as defined in [22, 12.14] and both map  $V_i$  to itself for each  $i \in [1, d]$ . By [22, 12.20 and 13.13(ii)], therefore, we can choose  $R \in \{T, T^\epsilon\}$  such that  $R$  is linked to the map  $(a, v) \mapsto a \cdot v$  at some point  $e \in X$  as defined in [22, 13.2]. By [22, 13.61],  $e \in X_{\text{sep}}$  and there exists  $r \in K^\times$  and  $s \in K$  such that  $R(v) = r\theta(e, v) + sv$  for all  $v \in L$ .  $\square$

**Proposition 3.15.** *Suppose that  $(K, L, q)$  is of type  $E_8$  and that*

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2)$$

*with  $q_2$  of type  $E_7$ ,  $q_1$  similar to the reduced norm of the quaternion division algebra representing  $\text{clif}(q_2)$  and  $1 \in L_2$ . Then the following hold:*

- (i) *There exists  $e \in X_{\text{sep}}$  such that  $\theta(e, L_i) = L_i$  for  $i = 1$  and  $2$ .*
- (ii) *Let  $e \in X$  be as in (i), let  $X_e$  be the subspace of  $X$  generated by elements of the form  $ev_1v_2 \cdots v_j$ , where  $v_i \in L_2$  for  $i \in [1, j]$  and  $j \geq 1$  is arbitrary, let  $\cdot_e, h_e$ , respectively,  $\theta_e$  denote the restriction of  $\cdot, h$ , respectively,  $\theta$  to  $X_e \times L_2, X_e \times X_e$ , respectively,  $X_e \times L_2$  and let*

$$\Xi_e = (K, L_2, q_2, 1, X_e, \cdot_e, h_e, \theta_e).$$

*Then  $\Xi_e$  is a quadrangular algebra.*

*Proof.* By 2.2, 2.4 and Proposition 2.6, both  $(q_1)_E$  and  $(q_2)_E$  are hyperbolic. We can thus choose  $V_i$  and  $q_i$  for  $i \in [1, d]$  as in Proposition 2.6(iii) with  $v_1 = 1$ ,  $V_i \subset L_2$  for  $i \in [1, 4]$  and  $V_i \subset L_1$  for  $i \in [5, 6]$ . By Proposition 3.14, therefore, there exists  $e \in X_{\text{sep}}$  such that  $\theta(e, L_i) = L_i$  for  $i = 1$  and  $2$ . Thus (i) holds.

Let  $\Xi_e$  be as described in (ii). To show that  $\Xi_e$  is a quadrangular algebra, it therefore suffices to show that  $X_e \cdot L_2 \subset X_e$ ,  $\theta(X_e, L_2) \subset L_2$  and  $h(X_e, X_e) \subset L_2$ . The first of these inclusions holds by the definition of  $X_e$ . To show the other two inclusions, we first choose non-zero elements  $v_i \in V_i$  for  $i \in [2, 5]$ . We can assume that  $e$  is the element of  $X$  chosen in [23, 6.4]. Thus the set  $1, v_2, \dots, v_5$  is  $e$ -orthogonal as defined in [23, 6.6]. By [23, 1.17(A3) and Prop. 6.16], there exists a non-zero  $v_6 \in L$  such that  $1, v_2, \dots, v_5, v_6$  is  $e$ -orthogonal and  $ev_2v_3v_4v_5v_6 = e$ . (We are not claiming that  $v_6 \in V_6$  or even  $v_6 \in L_1$ .) Let  $I_2$  be as in [23, 6.32], let  $J$  denote subset of  $I_2$  containing all the elements of  $I_2$  that are subsets of  $\{v_2, v_3, v_4\}$  together with the element  $\{v_5, v_6\} \in I_2$  (so  $|J| = 8$ ), let  $J_2$  be the elements of  $J$  of cardinality 2 (so  $|J_2| = 4$ ), let  $X_x$  for each  $x \in J$  be as in [23, 6.35], let  $M$  be the subspace of  $X$  spanned by  $\{X_m \mid m \in J\}$  and let  $N$  be the subspace of  $M$  spanned by  $\{X_m \mid m \in J_2\}$ . By [23, Prop. 6.34],  $\dim_K M = 16$  and  $M = eL_2 \oplus N$ . By [23, 6.37], we have  $M = X_e$ , by [23, Prop. 6.13], we have  $h(e, N) = 0$  and by [23, Props. 3.15 and 4.5(i)],  $h(e, eL_2) \subset L_2$  (since  $\theta(e, L_2) \subset L_2$ ). Hence  $h(e, X_e) \subset L_2$ . By repeated application of [23, 1.17(B1)–(B2)], it follows that  $h(X_e, X_e) \subset L_2$ . Since  $1 \in L_2$ , we have  $L_2^\sigma \subset L_2$ , where  $\sigma$  is as in 3.2. By repeated application of [23, 1.17(C3)–(C4)], it follows from  $\theta(e, L_2) \subset L_2$  first that  $\theta(eL_2, L_2) \subset L_2$  and then that  $\theta(X_e, L_2) \subset L_2$ . Thus (ii) holds.  $\square$

**Definition 3.16.** For each non-zero  $u$  in  $L$ , let  $\pi_u$  be the reflection of  $q$  given by

$$\pi_u(v) = f(u, v)u/q(u) - v$$

for all  $v \in L$ . Thus  $\pi_1 = \sigma$ , where  $\sigma$  is as in 3.2.

**Proposition 3.17.** Let  $H^\dagger$  and  $x_4$  be as in 3.5. Suppose that  $\varphi$  is a product of an even number of reflections of  $q$  as defined in 3.16. Then there exists an element  $h \in H^\dagger$  such that

$$x_4(v)^h = x_4(\varphi(v))$$

for all  $v$ .

*Proof.* Let  $u$  be a non-zero element of  $L$ . By [24, eq. (6)–(14)], there are elements  $w_1(0, q(u))$  and  $w_4(u)$  in  $H^\dagger$  such that

$$x_4(v)^{w_1(0, q(u))w_4(u)} = x_4(v/q(u))^{w_4(u)} = x_4(\pi_u \pi_1(v))$$

for each  $v \in L$ .  $\square$

**Notation 3.18.** Let  $M$  denote the subgroup of  $K^\times$  generated by the non-zero elements in the set  $\{q(\pi(a) + t) \mid (a, t) \in X \times K\}$ .

Thus

$$(3.19) \quad M \subset G(q) \cap \text{Hyp}_2(q)$$

by Proposition 3.8 and  $K^{\times 2} = \{q(\pi(a) + t) \mid (a, t) \in \{0\} \times K^\times\} \subset M$ .

**Proposition 3.20.** Let  $H^\dagger$  and  $x_4$  be as in 3.5. For each  $h \in H^\dagger$ , there exists a unique similitude  $\varphi_h$  of  $q$  such that

$$x_4(v)^h = x_4(\varphi_h(v))$$

for all  $v \in L$ . Furthermore, the map  $h \mapsto \gamma_h$  is a surjective homomorphism from  $H^\dagger$  to  $M$ , where  $\gamma_h$  is the multiplier of  $\varphi_h$ .

*Proof.* Let  $w_1(a, t)$  for non-zero  $(a, t) \in X \times K$  and  $w_4(u)$  for non-zero  $u \in L$  be as in [24, eqs. (6)–(7)]. Then

$$x_4(v)^{w_4(u)} = x_4(uf(u, v^\sigma) - q(u)v^\sigma)$$

and

$$x_4(v)^{w_4(a, t)} = x_4((\theta(a, v) + tv)/q(\pi(a) + t))$$

for all  $v \in L$  and all non-zero  $(a, t) \in X \times K$  by [24, eqs. (13)–(14)]. We have

$$q((\theta(a, v) + tv)/q(\pi(a) + t)) = q(v)/q(\pi(a) + t)$$

for all  $v \in L$  and all non-zero  $(a, t) \in X \times K$  (by 3.8) and

$$q(uf(u, v^\sigma) - q(u)v^\sigma) = q(v)q(u)^2$$

for all  $u, v \in L$  since  $q(v^\sigma) = q(v)$ . The claim holds, therefore, by [24, Thm. 2.1].  $\square$

**Proposition 3.21.** *If  $(K, L, q)$  is of type  $E_7$ , then  $G(q) = M$ .*

*Proof.* By (3.19), it suffices to show that  $G(q) \subset M$ . Let  $\varphi_1$  be a similitude of  $q$ , let  $x_4, U_4, H_0$  and  $H^\dagger \subset H_0$  be as in 3.5 and let  $h$  and  $\varphi$  be as in Proposition 3.7. Thus

$$x_4(v)^h = x_4(\varphi(v))$$

for each  $v \in L$  and  $\varphi$  is a similitude of  $q$  with the same multiplier as  $\varphi_1$ . Let  $H_1$  and  $H_2$  be the subgroups of  $H_0$  defined in [24, 3.12 and 3.14]. By [24, Thm. 3.15(ii)],  $H_0 = H_1H_2$  and by [24, Thm. 5.19],  $H_2 \subset H_1H^\dagger$ . We conclude that  $H_0 = H_1H^\dagger$ . By [24, Prop. 3.11],  $H_1$  centralizes  $U_4$ . There thus exists  $g \in H^\dagger$  such  $x_4(v)^g = x_4(\varphi(v))$  for each  $v \in L$ . The claim holds, therefore, by Proposition 3.20.  $\square$

**Proposition 3.22.** *If  $(K, L, q)$  is of type  $E_8$ , then  $G(q) = M$ .*

*Proof.* By (3.19), it suffices to show that  $G(q) \subset M$ . Let  $\varphi$  be a similitude of  $q$  whose multiplier is not in  $K^{\times 2}$ . By Proposition 3.10, it suffices to assume that  $\varphi$  is separable. Let

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2)$$

be the decomposition of  $q$  obtained by applying Proposition 2.16 to  $\varphi$ . Replacing  $\Xi$  by an isotope as defined in [23, 8.7], we can assume that the base point 1 lies in  $L_2$  (without changing the subgroup generated by the set of non-zero elements in  $\{q(\pi(a) + t) \mid (a, t) \in X \times K\}$ ). We can thus let  $e$  and

$$\Xi_e = (K, L_2, q_2, 1, X_e, \cdot_e, h_e, \theta_e)$$

with  $X_e \subset X$  be as in Proposition 3.15. By Proposition 3.21 (and the uniqueness assertion in Proposition 3.1), we conclude that  $\gamma$  is the product of elements in  $\{q(\pi(a) + t) \mid (a, t) \in X_e \times K\}$ .  $\square$

We can now prove Theorems 1.1 and 1.2. By Proposition 2.3, a quadratic form satisfying the hypotheses of Theorem 1.1 is of type  $E_7$  or  $E_8$ . By the existence assertion in Proposition 3.1, we can apply all the results in this section. Hence  $G(q) \subset \text{Hyp}_2(q)$  by (3.19) and Propositions 3.21 and 3.22. By [6, Thm. 20.14], we have  $\text{Hyp}_2(q) \subset G(q)$ . This concludes the proof of Theorem 1.1.

Suppose that  $(K, L, q)$  is of type  $E_8$  and that  $x_4, H_0$  and  $H^\dagger$  are as in 3.5. To prove Theorem 1.2, it suffices by Proposition 3.6 (and the existence assertion in

Proposition 3.1) to show that every element in  $H_0$  lies in  $H^\dagger$ . Let  $h \in H_0$ . By [24, eq. (19)], there is a similitude  $\varphi$  of  $q$  such that

$$x_4(v)^h = x_4(\varphi(v))$$

for all  $v \in L$ . Replacing  $h$  by a suitable element in  $hH^\dagger$ , we can assume, by Propositions 3.20 and 3.22, that  $\varphi$  is an isometry of  $q$  and hence a product of reflections of  $q$ . Again replacing  $h$  by a suitable element of  $hH^\dagger$ , we can assume, by Proposition 3.17 and [24, Prop. 3.16], that  $\varphi$  is the identity. By [24, Thm. 3.12],  $h = \alpha_u$  for some  $u \in C^\times$ , where  $C = K$  by [24, 3.6] and  $\alpha_u$  is as defined in [24, Prop. 3.11]. By [24, Prop. 3.13], it follows that  $h \in H^\dagger$ . This concludes the proof of Theorem 1.2.

#### 4. $R$ -EQUIVALENCE AND AN ALTERNATIVE PROOF OF THEOREM 1.2

In this section, we give an alternative proof of Theorem 1.2 based on Corollary 2.19 and various other results about  $R$ -equivalence. This proof is due to Skip Garibaldi. The methods employed in this section are completely different from those employed in the previous section; in particular, we make no further reference to the Moufang quadrangle  $\Gamma$  of §3.

Let  $G$  denote a reductive algebraic group of absolute type  $E_8$  whose Tits index over a field  $K$  is  $E_{8,2}^{66}$ . Our goal is to show that the group of  $K$ -rational points of  $G$  is generated by its root groups. By [9, 7.3], it suffices to assume that  $\text{char}(K) = 0$ . This will allow us to apply Corollary 2.19. By [9, 7.2], it suffices to show that  $G$  is  $R$ -trivial.

Now fix a maximal  $K$ -torus  $T$  containing a maximal  $K$ -split torus  $S$  in  $G$  and fix a pinning for  $G$  with respect to  $T$  over an algebraic closure of  $K$ . Number the simple roots  $\alpha_j$  as in [3, Chapter 6, Plate VII] and let  $\omega_i^\vee$  be the corresponding fundamental dominant co-weights, so  $\langle \alpha_j, \omega_i^\vee \rangle = \delta_{ij}$ . The fundamental co-weights  $\omega_1^\vee$  and  $\omega_8^\vee$  belong to the co-root lattice and so define cocharacters, in other words, homomorphisms from  $\mathbf{G}_m$  to  $T$ . Their images generate a subtorus  $S$  in  $T$  which is the connected component of the intersection (in  $T$ ) of the kernels of the roots  $\alpha_2, \dots, \alpha_7$ . There is a canonical isomorphism

$$(4.1) \quad \Phi : \bar{K}^\times \otimes_{\mathbb{Z}} T_* \xrightarrow{\sim} T(\bar{K}),$$

where  $T_*$  is the lattice of cocharacters of  $T$  and where  $\bar{K}$  is an algebraic closure of  $K$ ; see [18, 3.2.11]. Since the group  $G$  is of adjoint type, the  $\omega_i^\vee$  form a  $\mathbb{Z}$ -basis for  $T_*$ . Thus we may view  $\Phi$  as an isomorphism

$$\prod_{i=1}^8 \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_i^\vee \xrightarrow{\sim} T(\bar{K}).$$

This shows that the intersection of the kernels of the roots  $\alpha_2, \dots, \alpha_7$  is connected and that  $\Phi$  restricts to an isomorphism

$$(4.2) \quad (\bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_1^\vee) \times (\bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_8^\vee) \xrightarrow{\sim} S(\bar{K}).$$

The cocharacters  $\omega_1^\vee$  and  $\omega_8^\vee$  are defined over  $K$  by [2, Cor. 6.9], so (4.2) implies that  $S$  is  $K$ -isomorphic to the direct product of the images of the cocharacters  $\omega_1^\vee$  and  $\omega_8^\vee$ .

We next fix a parabolic  $P$  of  $G$  whose Levi subgroup is the connected reductive group  $Z_G(S)$ ; see [18, §13.4 and Lemma 15.1.2]. Let  $U$  be the unipotent radical

of  $P$  and let  $U^-$  be the unipotent radical of the opposite parabolic. The product  $U^- \times U$  is isomorphic as a variety to an affine space. By [1, Proof of Thm. 21.20], the natural map from  $G$  to  $G/P$  restricts to an isomorphism from  $U^-$  to an open subset of  $G/P$ . Hence the product map from  $U^- \times P$  to  $G$  defines an isomorphism from  $U^- \times P$  to an open subset of  $G$ . It follows that  $G$  is birationally equivalent to

$$U^- \times Z_G(S) \times U.$$

(This subvariety is the analog of the big cell for the Bruhat decomposition of  $G$  over  $K$ ; see [2, Prop. 4.10(d)].) We conclude that  $G$  is birationally equivalent to the product of  $Z_G(S)$  and an affine space.

Let  $H$  denote the derived subgroup of  $Z_G(S)$ . The sequence

$$1 \rightarrow S \rightarrow Z_G(S) \rightarrow H/(H \cap S) \rightarrow 1$$

is exact on  $L$ -points for every extension  $L/K$  because  $S$  is split. Hence  $Z_G(S)$  is birationally equivalent to the product of  $S$  with  $H/(H \cap S)$ .

The absolute Dynkin diagram of  $H$  is of type  $D_6$ . By [20, p. 211], the group  $H$  is  $\text{Spin}(q)$  for  $q$  a quadratic form over  $K$  with  $\dim q = 12$ ,  $\text{disc } q = 1$  and  $\text{clif}(q)$  split. As  $S$  centralizes  $H$ , the intersection  $H \cap S$  is contained in the center  $\mu_2 \times \mu_2$  of  $\text{Spin}(q)$ . We show that  $H \cap S$  is equal to the center of  $\text{Spin}(q)$ .

Since  $G$  is simply connected as well as adjoint, the co-roots  $\alpha_j^\vee$  provide also a  $\mathbb{Z}$ -basis for the cocharacter lattice  $T_*$ . Thus we may view the isomorphism  $\Phi$  in (4.1) as an isomorphism

$$\bar{K}^\times \otimes_{\mathbb{Z}} T_* = \prod_{i=1}^8 \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\alpha_i^\vee \xrightarrow{\sim} T(\bar{K}).$$

Then it follows from [18, 8.1.8] that  $\Phi$  restricts to an isomorphism

$$(4.3) \quad \prod_{i=2}^7 \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\alpha_i^\vee \xrightarrow{\sim} (H \cap T)^0(\bar{K}).$$

The expressions for the fundamental dominant weights  $\omega_i$  in terms of the roots  $\alpha_j$  in [3, Chapter 6, Plate VII] imply expressions for the fundamental dominant co-weights  $\omega_i^\vee$  in terms of the co-roots  $\alpha_j^\vee$ . These expressions yield

$$\omega_1^\vee(-1) = \alpha_2^\vee(-1)\alpha_3^\vee(-1) \quad \text{and} \quad \omega_8^\vee(-1) = \alpha_3^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1).$$

From (4.2) and (4.3), we see that these two elements both lie in  $S(\bar{K})$  and in  $(H \cap T)^\circ(\bar{K})$  and are nontrivial and distinct. We have thus produced two distinct nontrivial elements in  $(H \cap S)(\bar{K})$ . Hence  $H \cap S$  is, in fact, the entire center of  $\text{Spin}(q)$ .

Therefore  $H/(H \cap S)$  is  $\text{PGO}_+(q)$ . It follows by 2.19 that  $H/(H \cap S)$  is  $R$ -trivial. Therefore  $G$  is birationally equivalent to the product of  $\text{PGO}_+(q)$  times an affine space. Thus  $G$  itself is  $R$ -trivial by [4, p. 197, Cor.]. This concludes our second proof of Theorem 1.2.

We observe that this proof goes through verbatim for every group  $G$  of absolute type  $E_8$  in whose Tits index the roots  $\alpha_1$  and  $\alpha_8$  are circled. We conclude that for such groups,  $G$  is  $R$ -trivial and the group of  $K$ -rational points of  $G$  is generated by its root groups. With only minor modifications, the proof also shows that if  $G$  is adjoint of absolute type  $E_7$  with trivial Tits algebras and the root  $\alpha_1$  is circled in the Tits index of  $G$ , then  $G$  is  $R$ -trivial.

## 5. THEOREM 1.1 AND TRIALITY

In this section, we assume that  $\text{char}(K) \neq 2$ . We give an alternative proof of Theorem 1.1, based on completely different methods. We actually show:

**Proposition 5.1.** *Suppose the characteristic of the base field  $K$  is different from 2. If  $q$  is a quadratic form with trivial discriminant, then  $\text{Hyp}_2(q) = \text{Hyp}(q)$  in the following cases:*

- (i)  $\dim q = 8$  and the index of  $\text{clif}(q)$  is 1 or 2;
- (ii)  $\dim q = 12$  and  $\text{clif}(q)$  is split.

Since in each case  $G(q) = \text{Hyp}(q)$  by Proposition 2.15 and Corollary 2.17, Theorem 1.1 follows from Proposition 5.1.

We start with the case of 8-dimensional quadratic forms. If  $\text{clif}(q)$  is split, then  $q$  is a multiple of a 3-fold Pfister form, and the result follows from Lemma 2.14. Similarly, if  $q$  is isotropic, then  $q$  is Witt-equivalent to a multiple of a 2-fold Pfister form, and the result follows from Lemma 2.14. We may thus assume that  $(K, L, q)$  is of type  $E_7$  and let  $D$  be the quaternion division algebra over  $K$  that represents  $\text{clif}(q)$ . We show next that the Clifford algebra construction associates to  $q$  a skew-hermitian form  $h$  of rank 4 over  $D$ , and we shall complete the proof of Proposition 5.1(i) by proving that

$$\text{Hyp}(q) = \text{Sn}(h) = \text{Hyp}_2(q);$$

see Proposition 5.9.

Let  $(A, \sigma)$  be a central simple  $K$ -algebra of degree 8 with an orthogonal involution of trivial discriminant. The Clifford algebra  $C(A, \sigma)$  decomposes into a direct product of two central simple  $K$ -algebras of degree 8:

$$C(A, \sigma) = C_+(A, \sigma) \times C_-(A, \sigma).$$

Recall that  $C(A, \sigma)$  carries a canonical involution  $\underline{\sigma}$ , which induces orthogonal involutions  $\sigma_+$  and  $\sigma_-$  on  $C_+(A, \sigma)$  and  $C_-(A, \sigma)$  respectively. By triality (see [12, (42.3)]), the Clifford algebras of  $(C_+(A, \sigma), \sigma_+)$  and  $(C_-(A, \sigma), \sigma_-)$  satisfy

$$\begin{aligned} (C(C_+(A, \sigma), \sigma_+), \underline{\sigma_+}) &= (C_-(A, \sigma), \sigma_-) \times (A, \sigma), \\ (C(C_-(A, \sigma), \sigma_-), \underline{\sigma_-}) &= (A, \sigma) \times (C_+(A, \sigma), \sigma_+). \end{aligned}$$

**Proposition 5.2.** *The following hold:*

- (1) *If  $A$  is split, then  $(C_+(A, \sigma), \sigma_+)$  and  $(C_-(A, \sigma), \sigma_-)$  are isomorphic.*
- (2) *If  $(A, \sigma)$  is split and isotropic, then  $(C_+(A, \sigma), \sigma_+)$  and  $(C_-(A, \sigma), \sigma_-)$  are hyperbolic.*
- (3) *If  $(A, \sigma)$  is split and hyperbolic, then  $(C_+(A, \sigma), \sigma_+)$  and  $(C_-(A, \sigma), \sigma_-)$  are split and hyperbolic.*

*Proof.* (1) is well-known, (2) is in [12, (8.5)], and (3) follows from (2) and the fact that the Clifford invariant of a hyperbolic quadratic form is trivial.  $\square$

We apply this proposition in the following context: let  $(K, L, q)$  be an 8-dimensional quadratic space with  $\text{disc } q = 1$ , and assume  $\text{clif}(q)$  is represented by a quaternion division algebra  $D$ . Let  $\text{ad}_q: \text{End}_K L \rightarrow \text{End}_K L$  be the adjoint involution of  $q$ . We apply the discussion above with  $(A, \sigma) = (\text{End}_K L, \text{ad}_q)$ . Then  $C(A, \sigma) = C_0(L, q)$  and  $(C_+(A, \sigma), \sigma_+)$ ,  $(C_-(A, \sigma), \sigma_-)$  are isomorphic to  $(\text{End}_D W, \text{ad}_h)$  for



some 4-dimensional skew-hermitian space  $(W, h)$  over  $D$  (with its conjugation involution).

**Proposition 5.3.** *For an arbitrary extension  $E/K$ , the following statements are equivalent:*

- (a)  $q_E$  is hyperbolic;
- (b)  $D_E$  is split and  $(\text{End}_D W, \text{ad}_h)_E$  is hyperbolic;
- (c)  $D_E$  is split and  $(\text{End}_D W, \text{ad}_h)_E$  is isotropic.

*Proof.* (a)  $\Rightarrow$  (b): This readily follows from Proposition 5.2(3).

(b)  $\Rightarrow$  (c): Clear.

(c)  $\Rightarrow$  (a): This follows from Proposition 5.2(2) with  $(\text{End}_D W, \text{ad}_h)_E$  for  $(A, \sigma)$ ; then by triality  $(C_+(A, \sigma), \sigma_+)$  or  $(C_-(A, \sigma), \sigma_-)$  is isomorphic to  $(\text{End}_K L, \text{ad}_q)_E$ .

□

The next results 5.4–5.6 hold for skew-hermitian forms of arbitrary dimension.

**Lemma 5.4.** *Let  $(W, h)$  be a skew-hermitian space over a quaternion division algebra  $D$  over  $K$  and let  $E$  be a quadratic extension of  $K$ . If  $h$  is anisotropic, the following conditions are equivalent:*

- (i)  $E \cong K(h(v, v))$  for some  $v \in W$ ;
- (ii)  $D_E$  is split and  $h_E$  is isotropic.

*Proof.* If (i) holds, then  $E$  is isomorphic to a maximal subfield of  $D$ , hence  $D_E$  is split. Let  $h(v, v)^2 = a \in K^\times$ , so  $E \cong K(\sqrt{a})$ . Then  $v \cdot (h(v, v) + \sqrt{a}) \in W_E$  is isotropic for  $h_E$ . Thus, (ii) holds.

Conversely, if (ii) holds, then  $E$  is isomorphic to a maximal subfield of  $D$ . Let  $E = K(\sqrt{a})$  for some  $a \in K$ , and let  $\lambda \in D$  be a pure quaternion such that  $\lambda^2 = a$ . Suppose  $x + y\sqrt{a} \in W_E$  is  $h_E$ -isotropic for some  $x, y \in W$ . The condition  $h_E(x + y\sqrt{a}, x + y\sqrt{a}) = 0$  yields

$$h(x, x) + h(y, y)a = 0 \quad \text{and} \quad h(x, y) + h(y, x) = 0.$$

Since  $h$  is skew-hermitian, the second equation shows that  $h(x, y) \in K$ . Then

$$h(x + y\lambda, x + y\lambda) = 2h(x, y)\lambda - h(y, y)a - \lambda h(y, y)\lambda$$

and the right side commutes with  $\lambda$ . Therefore,  $h(x + y\lambda, x + y\lambda) = \lambda b$  for some  $b \in K^\times$ , and we have  $E \cong K(h(v, v))$  with  $v = x + y\lambda$ . □

For any skew-hermitian space  $(W, h)$  over a quaternion division algebra  $D$  over  $K$ , we let  $\text{Sn}(h)$  denote the group of spinor norms of  $h$ , which is the image of the Clifford group  $\Gamma(\text{End}_D W, \text{ad}_h) = \Gamma(W, h)$  under the multiplier map; see [12, (13.30)].

**Proposition 5.5.** *If  $h$  is anisotropic, then  $\text{Sn}(h) = \prod_E N(E/K)$ , where  $E$  runs over the quadratic extensions of  $K$  satisfying the equivalent conditions (i) and (ii) of Lemma 5.4.*

*Proof.* The multiplier map  $\Gamma(W, h) \rightarrow K^\times$  factors through the vector representation  $\Gamma(W, h) \rightarrow \text{O}_+(W, h)$ , where  $\text{O}_+(W, h)$  is the group of direct isometries of the space  $(W, h)$ . By [10, Thm. 6.2.17], this group is generated by transformations of the form

$$\tau_{v,r}: W \rightarrow W, \quad x \mapsto x - vh(vr, x)$$

where  $v \in W$  is an anisotropic vector and  $r \in D^\times$  satisfies  $r - \bar{r} = rh(v, v)\bar{r}$ . To compute the spinor norm of that transformation, observe that  $\tau_{v,r}$  is the identity on  $v^\perp$ , hence the spinor norm of  $\tau_{v,r}$  is the spinor norm of its restriction to the 1-dimensional subspace  $vD$ . Let  $\nu = h(v, v) \in D^\times$  and let  $h_v$  denote the restriction of  $h$  to  $vD$ , so

$$h_v(v\lambda, v\mu) = \bar{\lambda}\nu\mu \quad \text{for } \lambda, \mu \in D.$$

We have

$$\mathrm{O}_+(vD, h_v) = \{\theta \in K(\nu)^\times \mid \theta\bar{\theta} = 1\} \quad \text{and} \quad \Gamma(vD, h_v) = K(\nu)^\times$$

(where  $\theta \in K(\nu)^\times$  is identified with the map  $v\lambda \mapsto v\theta\lambda$  for  $\lambda \in D$ ). The vector representation  $\Gamma(vD, h_v) \rightarrow \mathrm{O}_+(vD, h_v)$  carries  $u \in K(\nu)^\times$  to  $u\bar{u}^{-1}$ , hence the spinor norm of that isometry is  $u\bar{u}K^{\times 2}$ ; see [12, (13.17)]. This shows that  $\mathrm{Sn}(h_v)$  consists of norms from the quadratic extension  $K(\nu)/K$ . Since  $\mathrm{Sn}(h)$  is generated by the groups  $\mathrm{Sn}(h_v)$  for the anisotropic vectors  $v \in W$ , the proposition follows.  $\square$

**Corollary 5.6.** *Let  $(W, h)$  be a skew-hermitian space over a quaternion division algebra  $D$  over  $K$ , and let  $p = \mathrm{char}(K) > 2$ . For  $\tilde{K} = K^{-p-\infty}$  the perfect closure of  $K$ , we have  $\mathrm{Sn}(h_{\tilde{K}}) \cap K = \mathrm{Sn}(h)$ .*

*Proof.* The inclusion  $\mathrm{Sn}(h) \subset \mathrm{Sn}(h_{\tilde{K}}) \cap K$  is clear, so it suffices to prove the reverse inclusion. Let  $x \in \mathrm{Sn}(h_{\tilde{K}}) \cap K$ . If  $x \in \tilde{K}^{\times 2}$ , then  $x \in K^{\times 2} \subset \mathrm{Sn}(h)$ . We may thus assume  $x \notin \tilde{K}^{\times 2}$ . By Proposition 5.5, there exist quadratic extensions  $\tilde{E}_1/\tilde{K}, \dots, \tilde{E}_r/\tilde{K}$  such that  $D_{\tilde{E}_i}$  is split and  $h_{\tilde{E}_i}$  is isotropic for each  $i \in [1, r]$ , and elements  $y_i \in \tilde{E}_i \setminus \tilde{K}$  for  $i \in [1, r]$  such that

$$(5.7) \quad x = N_{\tilde{E}_1/\tilde{K}}(y_1) \cdots N_{\tilde{E}_r/\tilde{K}}(y_r).$$

Let  $K' \subset \tilde{K}$  be the subfield generated by  $N_{\tilde{E}_1/\tilde{K}}(y_1), \dots, N_{\tilde{E}_r/\tilde{K}}(y_r)$  and, for  $i \in [1, r]$ , let  $E'_i = K'(y_i)$ . Thus,  $K'/K$  is a purely inseparable extension of finite degree, (5.7) yields

$$(5.8) \quad x = N_{E'_1/K'}(y_1) \cdots N_{E'_r/K'}(y_r),$$

and each  $E'_i/K'$  is a quadratic extension. For  $i \in [1, r]$ , let  $E''_i$  be the separable closure of  $K$  in  $E'_i$ ; it is a quadratic extension of  $K$  and we have

$$E'_i \cong E''_i \otimes_K K' \quad \text{and} \quad \tilde{E}_i \cong E''_i \otimes_K \tilde{K}.$$

Since quaternion division algebras do not split over extensions of odd degree, the condition that  $D_{\tilde{E}_i}$  is split shows that  $D_{E''_i}$  is split. Likewise, anisotropic skew-hermitian forms do not become isotropic over odd-degree extensions by [15, Thm. 3.5], hence  $h_{E''_i}$  is isotropic. Now, let  $[K' : K] = p^d$ ; taking the norm from  $K'$  to  $K$  of each side of (5.8), we obtain

$$x^{p^d} = N_{E'_1/K}(y_1) \cdots N_{E'_r/K}(y_r) = N_{E''_1/K}(N_{E'_1/E''_1}(y_1)) \cdots N_{E''_r/K}(N_{E'_r/E''_r}(y_r)).$$

Since  $x^{p^d} \equiv x \pmod{K^{\times 2}}$ , this equation shows that  $x$  is a product of norms from quadratic extensions over which  $D$  is split and  $h$  is isotropic, hence  $x \in \mathrm{Sn}(h)$  by Proposition 5.5.  $\square$

We now return to the context of Proposition 5.3. The following proposition completes the proof of Proposition 5.1(i):

**Proposition 5.9.** *For  $q$  an anisotropic 8-dimensional quadratic form and  $h$  the corresponding 4-dimensional skew-hermitian form as in Proposition 5.3, we have*

$$\text{Hyp}(q) = \text{Sn}(h) = \text{Hyp}_2(q).$$

*Proof.* Proposition 5.3 shows that the quadratic extensions  $E/K$  such that  $D_E$  is split and  $h_E$  is isotropic are exactly those such that  $q_E$  is hyperbolic, hence by Proposition 5.5 we have

$$\text{Sn}(h) = \text{Hyp}_2(q) \subset \text{Hyp}(q).$$

To complete the proof, we show  $\text{Hyp}(q) \subset \text{Sn}(h)$ . Let  $E/K$  be a finite-degree extension such that  $q_E$  is hyperbolic, let  $\tilde{K}$  be the perfect closure of  $K$  in some algebraic closure of  $E$ , and let  $K_1$  be the purely inseparable closure of  $K$  in  $E$ . The compositum  $E \cdot \tilde{K}$  of  $E$  and  $\tilde{K}$  satisfies  $E \cdot \tilde{K} \cong E \otimes_{K_1} \tilde{K}$ . Since  $q_E$  is hyperbolic,  $q$  is also hyperbolic over  $E \cdot \tilde{K}$ , hence  $D_{E \cdot \tilde{K}}$  is split and  $h_{E \cdot \tilde{K}}$  is isotropic, by Proposition 5.3. Therefore,  $\text{Sn}(h_{E \cdot \tilde{K}}) = (E \cdot \tilde{K})^\times$ . Since  $\tilde{K}$  is perfect, we may apply the norm principle for spinor norms (see [13, (6.2)]), which is a twisted analogue of Knebusch's norm theorem, to see that  $N(E \cdot \tilde{K}/\tilde{K}) \subset \text{Sn}(h_{\tilde{K}})$ . Since  $N(E/K_1) \subset N(E \cdot \tilde{K}/\tilde{K})$ , it follows that  $N(E/K_1) \subset \text{Sn}(h_{\tilde{K}})$ . Let  $p = \text{char}(K)$  if  $\text{char}(K) > 2$  and  $p = 1$  if  $\text{char}(K) = 0$ , so  $[K_1 : K] = p^d$  for some  $d \geq 0$ . For all  $x \in K_1$ , we have  $N_{K_1/K}(x) = x^{p^d}$ , hence

$$N(E/K) = N_{K_1/K}(N(E/K_1)) = N(E/K_1)^{p^d} \subset \text{Sn}(h_{\tilde{K}}).$$

But  $N(E/K) \subset K$ , hence Corollary 5.6 shows that  $N(E/K) \subset \text{Sn}(h)$ . Of course, we also have  $K^{\times 2} \subset \text{Sn}(h)$ , hence  $\text{Hyp}(q) \subset \text{Sn}(h)$ .  $\square$

Part (ii) of Proposition 5.1 follows from part (i) by the same arguments as in the proof of Corollary 2.17: let  $q$  be a 12-dimensional nondegenerate form with trivial discriminant and Clifford invariant. If  $q$  is isotropic, then it is Witt-equivalent to a 3-fold Pfister form and  $G(q) = \text{Hyp}_2(q)$  by Lemma 2.14. For the rest of the proof, suppose  $q$  is anisotropic, i.e.,  $q$  is of type  $E_8$ . Let  $\gamma \in G(q)$ . Since  $\text{char}(K) \neq 2$ , all similitudes of  $q$  are separable. We can thus fix a decomposition  $q = q_1 \perp q_2$  as in Proposition 2.16. Since  $\gamma \in G(q_2)$ , part (i) of Proposition 5.1 shows that  $\gamma \in \text{Hyp}_2(q_2)$ . Since  $\text{Hyp}_2(q_2) \subset \text{Hyp}_2(q)$  by Remark 2.18, it follows that  $\gamma \in \text{Hyp}_2(q)$ . This proves Proposition 5.1(ii).

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